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Abstract

The goal of this paper is to numerically investigate the effect of prestress on the propagation of elastic waves in straight and helical waveguides. The simulation technique, based on a semi-analytical finite element (SAFE) method, reduces the problem to the two-dimensional cross-section. The three-dimensional equilibrium equations of prestressed structures are written in the covariant and contravariant bases and used for the SAFE method. The eigenproblem can be solved by fixing the wavenumber k and finding the angular frequency ω or inversely. Dispersion curves for single straight and helical wires will be computed to quantify the effect of prestress on the wave propagation.

Introduction

Guided waves are commonly used to control large components such as plates, tubes and strands since they have the advantage to spread over long distances with little loss of energy. Because of the dispersive and multimodal behavior of guided waves, simulation becomes a very helpful tool for a proper analysis of these tests. The study of wave propagation in prestressed helical structures such as strands and springs requires the development of the equation of dynamics in a helical coordinate system. In the framework of modelling elastic guided waves, this paper presents a numerical approach dedicated to helical prestressed structures.

1 Curvilinear coordinate system

Let us considering a helical waveguide with a constant cross-section along the axis. The helix centerline curve can be described by the following position vector :

$$\mathbf{r}(s) = R\, cos(\frac{2\pi}{l}s)\mathbf{e}_X + R\, sin(\frac{2\pi}{l}s)\mathbf{e}_Y + (\frac{L}{l}s)\mathbf{e}_Z \quad (1)$$

where $l = \sqrt{L^2 + 4\pi^2\ R^2}$ is the curvilinear length of one helix step, $(\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z)$ represents the Cartesian basis. R and L are respectively the radius of the centerline in the $(\mathbf{e}_X, \mathbf{e}_Y)$ plane and the helix step along $\mathbf{e}_Z$ (see Figure1).

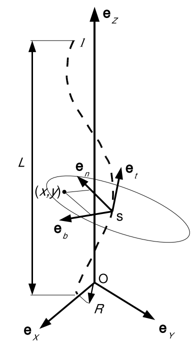

Figure 1: Helix centreline and its local basis

The variable s, corresponding to the arc length, varies between 0 and l. One considers the Serret-Frenet basis $(\mathbf{e}_n, \mathbf{e}_b, \mathbf{e}_t)$ associated to the helix centerline. The unit tangent, normal and binormal vectors to the centreline are respectively obtained from $\mathbf{e}_t = \partial\mathbf{r}/\partial s$, $\partial\mathbf{e}_t/\partial s = \kappa\mathbf{e}_n$ and $\partial\mathbf{e}_n/\partial s = \tau\mathbf{e}_b - \kappa\mathbf{e}_t$ ($\mathbf{e}_b = \mathbf{e}_t \wedge \mathbf{e}_n$). For a helix, both the curvature $\kappa = 4\pi^2 R/l^2$ and the tortuosity $\tau = 2\pi L/l^2$ are constant. Any cartesian vector $\mathbf{x}$ can be expressed in the Serret-Frenet basis as follows :

$$\mathbf{x}(x, y, s) = \mathbf{r}(s) + x\ \mathbf{e}_n(s) + y\ \mathbf{e}_b(s) \quad (2)$$

This mapping defines the non-orthonormal covariant basis $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$. $\mathbf{g}_1$, $\mathbf{g}_2$ and $\mathbf{g}_3$, are given by: $\mathbf{g}_1 = \partial\mathbf{x}/\partial x$, $\mathbf{g}_2 = \partial\mathbf{x}/\partial y$ and $\mathbf{g}_3 = \partial\mathbf{x}/\partial s$. We respectively define the contravariant basis $(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)$, which vectors are given by $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$. Components of the covariant (resp. contravariant) metric tensor are defined by $g_{mn} = \mathbf{g}_m \cdot \mathbf{g}_n$ (resp. $g^{mn} = \mathbf{g}^m \cdot \mathbf{g}^n$). This covariant metric tensor is given by:

$$\mathbf{g} = \begin{pmatrix} 1 & 0 & -\tau y \\ 0 & 1 & \tau x \\ -\tau y & \tau x & \tau^2(x^2+y^2) + (1-\kappa x)^2 \end{pmatrix} \quad (3)$$

Note that it does not depend on the third spatial variable s. The Christoffel symbol of the second kind, defined by $\Gamma_{ij}^k = \mathbf{g}_{i,j} \cdot \mathbf{g}^k$, are given by:

$$\begin{aligned}
\Gamma_{11}^k = \Gamma_{12}^k = \Gamma_{21}^k = \Gamma_{22}^k = 0, \Gamma_{23}^2 = \Gamma_{32}^2 = \Gamma_{23}^3 = \Gamma_{32}^3 = 0,\\
\Gamma_{33}^1 = \frac{\kappa(\tau y)^2}{1-\kappa x} + \kappa(1-\kappa x) - \tau^2 x, \Gamma_{33}^2 = -\frac{\kappa\tau^2 xy}{1-\kappa x} - \tau^2 y,\\
\Gamma_{33}^3 = \frac{\kappa\tau y}{1-\kappa x}, \Gamma_{13}^1 = \Gamma_{31}^1 = -\frac{\kappa\tau y}{1-\kappa x}, \Gamma_{23}^1 = \Gamma_{32}^1 = -\tau,\\
\Gamma_{13}^2 = \Gamma_{31}^2 = \frac{\kappa\tau x}{1-\kappa x} + \tau, \Gamma_{13}^3 = \Gamma_{31}^3 = -\frac{\kappa}{1-\kappa x}
\end{aligned} \tag{4}$$

All covariant (resp. contravariant) quantities are indexed down (resp. top). The notation $(\cdot)_{,i}$ $(i = 1, 2, 3)$ is used for derivatives with respect to x, y and s. For more details on curvilinear coordinate systems, the reader can refer to Ref. [1] for instance.

2 Variational formulation of prestressed structures

Assuming a linearly elastic material, static prestressed state with small strains, small superimposed perturbations with a time harmonic $e^{-i\omega t}$ dependence, the 3D variational formulation governing prestressed dynamics is given by [3]:

$$\int_{V_0} \delta\boldsymbol{\epsilon} : \mathbf{C} : \boldsymbol{\epsilon}\, dV_0 + \int_{V_0} tr(\nabla\delta\mathbf{u} \cdot \boldsymbol{\sigma}_0 \cdot (\nabla\mathbf{u})^T)\, dV_0 - \omega^2 \int_{V_0} \rho\delta\mathbf{u} \cdot \mathbf{u}\, dV_0 = 0 \tag{5}$$

for any kinematically admissible trial field $\delta\mathbf{u}$. $\mathbf{u}$ and $\boldsymbol{\epsilon}$ respectively denote the displacement vector and the Green-Lagrange strain tensor. ρ is the material density and V_0 represents the prestressed structural volume. $\mathbf{C}$ and $\boldsymbol{\sigma}_0$ denote respectively the matrix of material properties and the Cauchy prestressed tensor (assumed to be known). $tr(\cdot)$ and $\nabla(\cdot)$ respectively denote the trace and the gradient tensor with respect to predeformed variables. Each quantity of Eq. (5) must be written in the covariant and contravariant coordinate systems. In the covariance basis, components of $\nabla\mathbf{u}$, denoted γ_{ij}, can be written as follows: $\gamma_{ij} = (u_{i,j} + u_{j,i}) - \Gamma_{ij}^k u_k$. The strain tensor $\boldsymbol{\epsilon}$ is written in terms of the gradient tensor as follows: $\boldsymbol{\epsilon} = 1/2(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$.

3 Semi analytical finite element method

In order to still speak of propagation modes, the application of a SAFE method relies on a spatial Fourier transform in the s direction by assuming an e^{iks} separation. Three criteria are implicitly needed. We assumed that the cross-section of the guide does not vary along s nor the material properties. The third criterion is related to the variational formulation and requires that the coefficients of every differential operator do not depend on s, which is true if the metric tensor does not depend on s either. As shown previously, this occurs for the metric tensor defined by Eq. (3). A SAFE method can thus be applied (see Ref. [2] for more details on unprestressed structures). The displacement vector and its trial field are rewritten as: $\mathbf{u} = \mathbf{u}(x,y)e^{i(ks-\omega t)}, \delta\mathbf{u} = \delta\mathbf{u}(x,y)e^{-i(ks-\omega t)}$. e^{iks} are separated from all field components, and $\partial/\partial s$ replaced by ik, where k denotes the axial wavenumber.

Then, the FE discretization of Eq. (5) finally arrives at the following eigenvalue problem for the column vector $\mathbf{U}$ containing displacement degrees of freedom:

$$\{\mathbf{K}_1 - \omega^2\mathbf{M} + ik(\mathbf{K}_2 - \mathbf{K}_2^T) + k^2\mathbf{K}_3\}\mathbf{U} = \mathbf{0} \tag{6}$$

The eigenproblem obtained by fixing ω and finding k is quadratic. At fixed real k, the eigenproblem given by Eq. (6) is linear for finding ω^2. This approach, only valid in the absence of damping, allows to find propagating (k purely real), evanescent (k purely imaginary) and inhomogeneous waves (k fully complex).

Conclusion

A SAFE method is proposed to analyse the effect of prestress on wave propagation in straight and helical waveguides. This SAFE method is based on the equation of prestressed structural dynamics, which are written in a curvilinear helical coordinate system. Results will be presented during the conference and checked with some reference solutions. The effect of prestress on the propagation of guided waves will be outlined.

References

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